MODULE 4

Linear system of equations

A linear system of m equations in n unknowns $x_1, \dots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots \dots \dots \dots &\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \quad \text{— (1)}$$

* The system is called linear because each variable x_j appears in the first power only, just as in the equation of a straight line.
* $a_{11}, \dots, a_{mn}$ are given numbers called the coefficients of the system.
* $b_1, \dots, b_m$ on the right are also given numbers. If all b_j are zero, then (1) is called a homogeneous system. If atleast one b_j is not zero, then (1) is called nonhomogeneous system.

$\left.\begin{aligned} x^2+y^2 &= 25 \\ xy+2x &= 0 \end{aligned}\right\}$ (not linear)

* A solution of (1) is a set of numbers $x_1, \dots, x_n$ that satisfies all the m equations.
* A solution vector of (1) is a vector x whose components form a solution of (1). If the system (1) is homogeneous, it always has atleast the trivial solution $x_1 = 0, \dots x_n = 0$.

Matrix Form of the Linear system (1) is,

$$Ax = b$$

where the coefficient Matrix $A = [a_{jk}]$ is the $m \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

$$\text{and } b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

are column vectors. we assume that A is not a zero matrix.

* The matrix,

$$\tilde{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \\ \cdot & & & \\ a_{m1} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix of the system (1).

Geometric interpretation, Existence and uniqueness of solutions.

If $m = n = 2$, we have two equations in two unknowns, x_1, x_2.

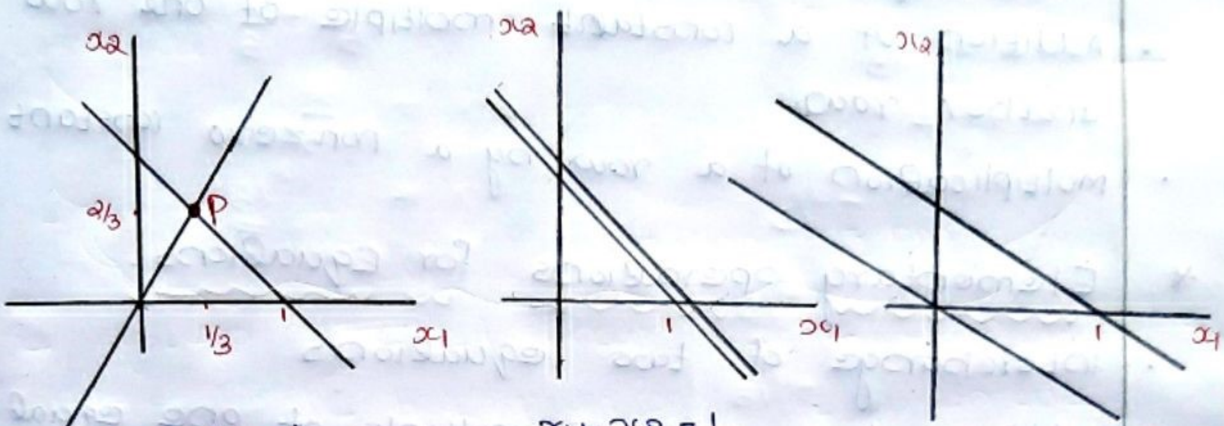
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

If we interpret x_1, x_2 as coordinates on the x_1x_2-plane, then each of the 2 equations represents a straight line, and (x_1, x_2) is a solution iff the point

P with coordinates x_1, x_2 lies on both lines.
Hence there are 3 possible cases.

a) precisely one solution if the lines intersect
b) infinitely many solutions if the lines coincide
c) No solution if the lines are parallel.

for eg,



$x_1 + x_2 = 1$	$x_1 + x_2 = 1$	$x_1 + x_2 = 1$
$2x_1 - x_2 = 0$	$2x_1 + 2x_2 = 2$	$x_1 + x_2 = 0$
case (a)	case (b)	case (c)

Gauss Elimination and Back Substitution.

Eg: $2x_1 + 5x_2 = 2$ ——— (1)

$13x_2 = -26$

A. $x_2 = -\frac{26}{13} = -2.$

Sub $x_2 = -2$ in (1)

$2x_1 - 10 = 2$

$2x_1 = 12$

$x_1 = 6$ //

or

The augmented matrix for the given system is,

$$\begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$$

Elementary Row operations. Row - Equivalent Systems.

* **Elementary Row operations for matrices**

- Interchange of two rows
- Addition of a constant multiple of one row to another row.
- multiplication of a row by a nonzero constant c.

* **Elementary operations for Equations.**

- Interchange of two equations
- Addition of a constant multiple of one equation to another equation.
- multiplication of an equation by a nonzero constant c

Row - Equivalent systems.

Row - equivalent linear systems have the same set of solutions.

* A system is called consistent if it has atleast one solution, but inconsistent if it has no solutions at all.

? Solve the linear system,

$$2x + y + 2z = 7$$

$$x - 2y + z = 8$$

$$3x + y - z = 9$$

A). The Augmented matrix for the system is given by,

$$\tilde{A} = \begin{bmatrix} 2 & 1 & 2 & 7 \\ 1 & -2 & 1 & 8 \\ 3 & 1 & -1 & 9 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{2} R_1$

$R_3 \to \frac{1}{3} R_3$

$$\approx \begin{bmatrix} 2 & 1 & 2 & 7 \\ 0 & -5/2 & 0 & 9/2 \\ 1 & 1/3 & -1/3 & 9/3 \end{bmatrix}$$

$R_2 \to 2 R_2$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 7 \\ 0 & -5 & 0 & 9 \\ 1 & 1/3 & -1/3 & 3 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 7 \\ 0 & -5 & 0 & 9 \\ 0 & -1/6 & -4/3 & -1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 7 \\ 0 & -5 & 0 & 9 \\ 0 & -1/6 & -4/3 & -1/2 \end{bmatrix}$$

$R_1 \to R_1 + \frac{1}{5} R_2$

$R_3 \to R_3 - \frac{1}{30} R_2$

$$\sim \begin{bmatrix} 2 & 0 & 2 & \frac{44}{5} \\ 0 & -5 & 0 & 9 \\ 0 & 0 & -4/3 & -4/5 \end{bmatrix}$$

$R_3 \to -\frac{1}{4} R_3$

$$\sim \begin{bmatrix} 2 & 0 & 2 & 44/5 \\ 0 & -5 & 0 & 9 \\ 0 & 0 & 1/3 & 1/5 \end{bmatrix}$$

$\frac{1}{3}z = \frac{1}{5}$

$5z = 3$

$z = 3/5$

$-5y = 9$

$y = -9/5$

$2x + \frac{6}{5} = \frac{44}{5}$

$2x = \frac{44}{5} - \frac{6}{5}$

$2x = \frac{38}{5}$

$x = 19/5$

$$X = \begin{bmatrix} 19/5 \\ -9/5 \\ 3/5 \end{bmatrix}$$

2. Solve the linear system,

$3x - 3y - 2z = 6$

$2x - 4y - 3z = 8$

$-3x + 6y + 8z = -5$

A). $$\tilde{A} = \begin{bmatrix} 3 & -3 & -2 & 6 \\ 2 & -4 & -3 & 8 \\ -3 & 6 & 8 & -5 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 3 & -3 & -2 & 6 \\ 2 & -4 & -3 & 8 \\ -3 & 6 & 8 & -5 \end{bmatrix}$$

$R_2 \to \frac{1}{2} R_2$

$R_3 \to -\frac{1}{3} R_3$

$$\tilde{A} \sim \begin{bmatrix} 3 & -3 & -2 & 6 \\ 1 & -2 & -3/2 & 4 \\ 1 & -2 & -8/3 & +5/3 \end{bmatrix}$$

$R_2 \to 3R_2 - R_1$

$R_3 \to 3R_3 - R_1$

$$\sim \begin{bmatrix} 3 & -3 & -2 & 6 \\ 0 & -3 & -5/2 & 6 \\ 0 & -3 & -6 & -1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$R_3 \to R_3 - R_2$.

$$\sim \begin{bmatrix} 3 & 0 & 1/2 & 0 \\ 0 & -3 & -5/2 & 6 \\ 0 & 0 & -7/2 & -7 \end{bmatrix}$$

$-7/2\, z = -7$

$-7z = -14$

$z = 2$

$-3y - 5/2 \times 2 = 6$

$-3y - 5 = 6$

$-3y = 11$

$y = -11/3$

$3x + \frac{2}{2} = 0$

$3x + 1 = 0$

$3x = -1$

$x = -1/3$

$$x = \begin{bmatrix} -1/3 \\ -11/3 \\ 2 \end{bmatrix}$$

2. $x + 2y - 3z = 1$

$2x + 5y - 8z = 4$

$3x + 8y - 13z = 7$

A). The Augmented matrix

$$\tilde{A} = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 2 & 5 & -8 & 4 \\ 3 & 8 & -13 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -4 & 4 \end{bmatrix}$$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $R_1 \to R_1 - 2R_2$, $R_3 \to R_3 - 2R_2$

here one variable is free variable.

$x + 2y - 3z = 1$ —(1)

$y - 2z = 2.$

$y = 2z + 2$

$z = k,$

$y = 2k + 2.$

$x = -2y + 3z$

$= -2(2k + 2) + 3k.$

$x = -4k - 4 + 3k$

$= -k - 4$

$= -(k + 4)$

$$\begin{bmatrix} -(k+4) \\ 2k+2 \\ k \end{bmatrix}$$

This system is consistent.

2. Determine the following system is consistent.

$x_2 - 4x_3 = 8$

$2x_1 - 3x_2 + 2x_3 = 1$

$4x_1 - 8x_2 + 12x_3 = 1$

A) $\tilde{A} = \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix}$ $R_3 \to R_3 - 2R_1$

$\sim \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix}$ $R_3 + 2R_2$

$\sim \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}$

$2x_1 - 3x_2 + 2x_3 = 1$

$x_2 - 4x_3 = 8$

$0x_1 + 0x_2 + 0x_3 = 15$

$0 = 15$

$\therefore$ No solution.

$\therefore$ The system is inconsistent.

3. Find the rank of the matrix.

$\begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 4 & 0 & 2 & 0 \end{bmatrix}$

A

$$\begin{bmatrix} 4 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 \\ 4 & 0 & 2 & 0 \end{bmatrix}$$

$R_2 \to R_1/2$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 4 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{echelon form.}$$

Rank = 2 //

2 Investigate, for what values of λ and μ do the system of equations

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

i) no solution?

ii) unique solution?

iii) an infinity of solutions?

A).

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

unique solution

~~$\lambda - 1 = 0$~~ $\quad \lambda - 3 \neq 0$

~~$\lambda =$~~ $\quad \lambda \neq 3$

no solution

$\lambda-3=0$ & $\mu-10\neq 0$

$\lambda=3$ and $\mu\neq 10$

an infinite solution.

$\lambda=3$ & $\mu=10$

Row Echelon Form and information from it

* At the end of the Gauss elimination the form of the coefficient matrix, the augmented matrix and the system itself are called the row echelon form.

* ~~The rank of a matrix is the number of non zero rows on any row-echelon matrix to which~~ row operations.

A ~~can be carried by elementary~~

~~for all non-zero rows the~~

* A matrix is in reduced row-echelon form if it is in row-echelon form and every pivot is the only non-zero entry in its column.

* Let A be an m×n matrix and B be its row-echelon form. The rank of A is the maxi no. of pivots of B.

non zero rows

* Given the linear system $AX=B$, and the augmented matrix $(A|B)$

If $\text{rank}(A)=\text{rank}(A|B)=n$ (no. of variables in the system) then the system has a unique solution.

If $\text{rank}(A)=\text{rank}(A|B)<n$, then the system has ∞-many solutions.

If $\text{rank}(A)<\text{rank}(A|B)$, then the system is inconsistent ie, no solution.

Homogeneous Linear System

The linear system,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

is called homogeneous if all the b_j's are zero, and nonhomogeneous if one or several b_j's are not zero.

Theorem:

A homogeneous linear system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \qquad \text{——— (2)}$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

$$Ax = 0$$

always has the trivial solution $x_1 = 0, \ldots, x_n = 0$. Nontrivial solution exist iff rank $A < n$. If rank $A = r < n$ together with $x = 0$, form a vector space of dimension $n - r$ called the solution space of (2).

In particular, if $x_{(1)}$ and $x_{(2)}$ are solution vector of (4), then $x = c_1 x_{(1)} + c_2 x_{(2)}$ with any scalars c_1 and c_2 is a solution vector of (4).

Theorem 2. Homogeneous Linear system with Fewer Equations than unknowns

A homogeneous linear system with fewer equations than unknowns always has nontrivial solutions.

Note:

If rank $(A) = n$, (where n is the no. of variables) which leads to the conclusion that we have one distinct solution, which would be the trivial $(0,0,0)$.

If rank $(A) < n \Rightarrow$ infinitely many solutions, in addition to the trivial solution.

Rank of a matrix using determinant

The rank of a matrix is the order of the largest square submatrix whose determinant is not zero.

For a homogeneous linear system, $AX=0$,
If $|A| \neq 0$ then unique solution and
If $|A| = 0$ infinitely many solution.

Linear independence and dependence of vectors

* A set $\{x_1, x_2, \ldots x_r\}$ of r n-vectors is said to be a linearly dependent set, if there exists r scalars $k_1, k_2, \ldots k_r$, not all zero, such that

$$k_1 x_1 + k_2 x_2 + \ldots + k_r x_r = 0$$

where 0 denotes the n-vector with all its components zero.

* A set $\{x_1, x_2, \ldots, x_r\}$ of r n-vectors is said to be a linearly independent set, if it is not a linearly dependent set ie, if for any r scalars $k_1, k_2, \ldots, k_r$,

$$k_1 x_1 + k_2 x_2 + \ldots + k_r x_r = 0 \Rightarrow k_1 = k_2 = \ldots k_r = 0,$$

where O denotes the n-vector with all its components zero.

where 0 denotes the n-vector with all its components zero.

? Check whether the matrix $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & 5 \end{bmatrix}$ is linearly independent or not ?

A). First find the row-reduced echelon form,

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} . \; R_2 \to R_2 - 2R_1 \quad R_3 \to R_3 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$\therefore$ rank = 2

linearly dependent.

~~or~~

~~First the~~

2 check whether the matrix $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & 5 \end{bmatrix}$ is linearly independent or not. and find a relation connecting them.

A). for finding the relation

Let k_1, k_2, k_3 are scalars,

$$k_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + k_3 \begin{pmatrix} 4 \\ 5 \\ 5 \end{pmatrix} = 0$$

$\Rightarrow$ $k_1 + 2k_2 + 4k_3 = 0$

$k_1 + 3k_2 + 5k_3 = 0$

$2k_1 + k_2 + 5k_3 = 0.$

The corresponding matrix is given by,

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$.

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$k_1 + 2k_2 + 4k_3 = 0$

$k_2 + k_3 = 0$

$k_2 = -k_3$

$k_1 + -2k_3 + 4k_3 = 0$

$k_1 + 2k_3 = 0$

$k_1 = -2k_3$.

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -2k_3 \\ -k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ we get, $-2v_1 \underline{-v_2} + v_3 = 0$

A vector as a Linear combination of a set of vectors

A vector x which can be expressed in the form

$$x = k_1x_1 + k_2x_2 + \cdots + k_rx_r.$$

where $k_1, k_2, \ldots k_r$ are scalars, is said to be a linear combination of the set $\{x_1, x_2, \cdots, x_r\}$ of vectors.

2 Express $V = (1, -2, 5)$ in R^3 as a linear combination of the vectors.

$u_1 = (1,1,1)$ $u_2 = (1,2,3)$ $u_3 = (2,-1,1)$

A). $k_1u_1 + k_2u_2 + k_3u_3 = V$

$$k_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + k_3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$\Rightarrow$ $k_1 + k_2 + 2k_3 = 1$

$k_1 + 2k_2 - k_3 = -2$

$k_1 + 3k_2 + k_3 = 5$

$$\begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 1 & 2 & -1 & \vdots & -2 \\ 1 & 3 & 1 & \vdots & 5 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

$k_1 + k_2 + 2k_3 = 1$

$k_2 - 3k_3 = -3$

$5k_3 = 10$

$k_3 = 2$ //

$k_2 - 3\times 2 = -3$

$k_2 - 6 = -3$

$k_2 = -3+6 = 3$ //

$k_1 + 3 + 4 = 1$

$k_1 = 1 - 7 = -6$ //

$-6u_1 + 3u_2 + 2u_3 = v$

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

$K_1 + K_2 + 2K_3 = 1$

$K_2 - 3K_3 = -3$

$5K_3 = 10$

$K_3 = 2$ //

$K_2 - 3\times 2 = -3$

$K_2 - 6 = -3$

$K_2 = -3+6 = 3$ //

$K_1 + 3 + 4 = 1$

$K_1 = 1 - 7 = -6$ //

$$-6U_1 + 3U_2 + 2U_3 = V$$

2 Show that the functions $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = t$ from R to R are linearly independent?

A). Let V_1, V_2, V_3 vectors are said to be linearly independent if,

$$K_1V_1 + K_2V_2 + K_3V_3 = 0 \Rightarrow K_1 = 0, K_2 = 0, K_3 = 0$$

here, $V_1 = \sin t$, $V_2 = \cos t$, $V_3 = t$

$$K_1 \sin t + K_2 \cos t + K_3 t = 0$$

In particular, $t = 0$,

$$K_1 \sin 0 + K_2 \cos 0 + K_3 \times 0 = 0$$

$K_2 = 0$

when $t = \pi$,

$k_1 \sin\pi + \cancel{k_2 \cos\pi} + k_3\pi = 0$

$0 - K_2$

$K_1 \sin\pi + K_3\pi = 0$

$K_3\pi = 0$

$K_3 = 0$ //

we get, $K_2 = K_3 = 0$,

$K_1 \sin t = 0$

$\Rightarrow t = \pi/2$

$\Rightarrow K_1 \times 1 = 0 = K_1 = 0.$

$\therefore$ $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = t$ from $R \to R$ are linearly independent.

2 suppose the vectors u, v, w are linearly independent. show that the vectors $u+v$, $u-v$, $u-2v+w$ are also linearly independent.

A). Given that u, v, w are linearly independent, so we can write as,

$K_1 u + K_2 V + K_3 w = 0 \Rightarrow K_1 = 0, K_2 = 0, K_3 = 0$ ——— (1)

we've to S.T $u+v$, $u-v$, $u-2v+w$ are also linearly independent ie, we've to show,

$c_1(u+v) + c_2(u-v) + c_3(u-2v+w) = 0 \Rightarrow c_1 = 0, c_2 = 0, c_3 = 0.$

$c_1(u+v) + c_2(u-v) + c_3(u-2v+w) = 0$

$= c_1 u + c_2 u + c_3 u + c_1 v + c_2 \times -v - 2c_3 v + c_3 w = 0.$

$u\underbrace{(c_1 + c_2 + c_3)}_{\text{constant}} + v\underbrace{(c_1 - c_2 - 2c_3)}_{\text{constant}} + c_3 w = 0.$

$\Rightarrow c_i u + c_j v + c_3 w = 0$ ——— (2) $\quad (c_i = c_1 + c_2 + c_3$, $c_j = c_1 - c_2 - 2c_3)$

comparing (1) and (2) we get,

$c_1 = 0, c_2 = 0, c_3 = 0.$

$c_1 + c_2 + c_3 = 0$ ——— (3)

$c_1 - c_2 + 2c_3 = 0$ ——— (4).

$c_3 = 0$ in (3) & (4), we get,

$$\left.\begin{array}{l} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{array}\right\} \Rightarrow c_1 = 0, c_2 = 0$$

$c_1(u+v) + c_2(u-v) + c_3(u-2v+w) = 0 \Rightarrow c_1 = 0, c_2 = 0, c_3 = 0.$

$\therefore$ linearly independent.

Matrix Eigen value problems

* A matrix eigenvalue problem considers the vector equation

$$AX = \lambda X \quad \text{——— (1)}$$

Here A is a given square matrix, λ an unknown scalar, and x an unknown vector.

* The λ's that satisfy (1) are called eigenvalues of A and the corresponding nonzero x's that also satisfy (1) are called eigen vectors of A. (characteristic vectors of A)

* Let A be an $n \times n$ square matrix and λ an indeterminate. Then the matrix $A - \lambda I$, where I is a unit matrix of order n, is called the characteristic matrix of A.

* The determinant $|A - \lambda I|$ is a non-null polynomial of degree n in λ and is called characteristic polynomial of A.

* The equation $|A - \lambda I| = 0$ is called the characteristic eqn of A and its roots are called the characteristic roots or latent roots or eigen values of A.

Definition:

Any non-zero vector X is said to be a eigen vector of a matrix A, if there exists a number λ such that $AX=\lambda X$

Determination of characteristic vectors

The equation $AX=\lambda X$ can be written as, $(A-\lambda I)X=0$ —(2)

* The necessary and sufficient condition for the equation (2) to possess a non-zero solution $(X\neq 0)$ is that the determinant of the coefficient matrix, ie, $|A-\lambda I|$ is zero.

* ∴ if X is a eigen vector of the matrix A, the corresponding value of λ satisfies the characteristic eqn $|A-\lambda I|=0$ and hence is a eigen value of A.

? Determine the eigen values of the matrix, $A=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}$

A) $A-\lambda I=\begin{bmatrix}-5-\lambda & 2\\ 2 & -2-\lambda\end{bmatrix}$

$|A-\lambda I| = (-5-\lambda)(-2-\lambda)-4$

$= 10+5\lambda+2\lambda+\lambda^2-4$

$= \lambda^2+7\lambda+6$

$|A-\lambda I|=0$

$\Rightarrow \lambda^2+7\lambda+6=0$

$\Rightarrow \lambda = -6, -1$ $\Rightarrow$ eigen values.

2. $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$

A) $A-\lambda I=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & -\lambda\end{bmatrix}$

$|A-\lambda I| = (-2-\lambda)[(1-\lambda)-\lambda-12] - 2[-2\lambda-6] + -3[-4+1-\lambda]$

$= (-2-\lambda)[-\lambda+\lambda^2-12] + 4\lambda+12+9+3\lambda$

$$= 2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 7\lambda + 21$$

$$= -\lambda^3 - \lambda^2 + 21\lambda + 45$$

$$= -\lambda^3 - \lambda^2 + 21\lambda + 45$$

$$|A - \lambda I| = 0 \Rightarrow \quad -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\lambda = 5,\ \lambda = -3,\ -3$$

$$\lambda = 5,\ \lambda = -3 \quad \Rightarrow \text{eigen values.}$$

2.

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$$

A).

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -1 & 1 \\ 7 & -5-\lambda & 1 \\ 6 & -6 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (3-\lambda)\left[(-5-\lambda)(2-\lambda) + 6\right] + 1\left[14 - 7\lambda - 6\right] + 1\left[-42 - (30 - 6\lambda)\right]$$

$$= (3-\lambda)\left[-10 + 5\lambda - 2\lambda + \lambda^2 + 6\right] - 7\lambda + 8 - 42 + 30 + 6\lambda$$

$$= (3-\lambda)\left[\lambda^2 + 3\lambda - 4\right] - \lambda - 4$$

$$= 3\lambda^2 + 9\lambda - 12 - \lambda^3 - 3\lambda^2 + 4\lambda - \lambda - 4$$

$$= -\lambda^3 + 12\lambda - 16$$

$$|A - \lambda I| = 0 \Rightarrow \quad \lambda^3 - 12\lambda + 16 = 0$$

$$\lambda = -4,\ \lambda = 2 \quad \Rightarrow \text{eigen values.}$$

or.

$\lambda = 2$ is a solution.

$\lambda - 2 = 0$. To find other solutions, divide $\lambda^3 - 12\lambda + 16$ by $\lambda - 2$

$$\lambda - 2 \overline{\big) \lambda^3 - 12\lambda + 16} \quad \lambda^2 + 2\lambda - 8$$

$$\lambda^3 - 2\lambda^2$$

$$2\lambda^2 - 12\lambda + 16$$

$$2\lambda^2 - 4\lambda$$

$$-8\lambda + 16$$

$$-8\lambda + 16$$

$$0$$

$$\lambda^3 - 12\lambda + 16 = (\lambda - 2)(\lambda^2 + 2\lambda - 8)$$

$$\therefore \lambda = 2, 2, -4$$

2 Find the eigen values for the following matrices.

1. $\begin{bmatrix} 3.0 & 0 \\ 0 & -0.6 \end{bmatrix}$

A). $A - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 0 & -0.6-\lambda \end{bmatrix}$

$|A - \lambda I| = \cancel{[3-\lambda \times [-0.6-\lambda]]}\ (3-\lambda)(-0.6-\lambda) - 0$

$= \cancel{3-\lambda+0.6+\lambda}$ $\quad = -3\times 0.6 - 3\lambda + 0.6\lambda + \lambda^2$

$= 3$ $\quad = -1.8 - 3\lambda + 0.6\lambda + \lambda^2$

$|A - \lambda I| = 0 \Rightarrow \lambda^2 - 2.4\lambda - 1.8 = 0$

$\lambda = 3, \lambda = -0.6 \Rightarrow$ eigen values.

2. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

A) $\lambda = 0$ is the eigen value.

3. $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$

A $A - \lambda I = \begin{bmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{bmatrix}$

$|A - \lambda I| = (5-\lambda)(-6-\lambda) + 18$

$= -30 - 5\lambda + 6\lambda + \lambda^2 + 18$

$\lambda^2+\lambda-12$

$|A-\lambda I|=0 \Rightarrow \lambda^2+\lambda-12=0$

$\lambda=-4,\ \lambda=3. \Rightarrow$ eigon value

4). $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

A). $A-\lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$

$|A-\lambda I| = (1-\lambda)(4-\lambda)-4$

$= 4-\lambda-4\lambda+\lambda^2-4$

$= \lambda^2-5\lambda$

$= \lambda(1-5\lambda)$

$|A-\lambda I|=0 \Rightarrow \lambda(1-5\lambda)=0$

$\lambda=0$ or $(1-5\lambda)=0$

$5\lambda=1$

$\lambda=1/5$

eigon values 0, 1/5

5. $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

A). $A-\lambda I = \cancel{3[4]-5[0]+3[0]}$

$= \cancel{12} \begin{bmatrix} 3-\lambda & 5 & 3 \\ 0 & 4-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

$= (3-\lambda)[(4-\lambda)(1-\lambda)]$

$= (3-\lambda)[4-4\lambda-\lambda+\lambda^2]$

$= (3-\lambda)[\lambda^2-5\lambda+4]$

$= 3\lambda^2-15\lambda+12-\lambda^3+5\lambda^2-4\lambda$

$= -\lambda^3+8\lambda^2-19\lambda+12$

$|A-\lambda I|=0 \Rightarrow \lambda^3-8\lambda^2+19\lambda-12=0.$

eigen values, $\lambda = 1, 4, 3$

2. $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1/2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

A) $A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & -1 \\ 0 & 1/2-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{bmatrix}$

$|A-\lambda I| = (2-\lambda)[(1/2-\lambda)(4-\lambda)] - 1[-1/2+\lambda]$

$= (2-\lambda)[(2-\lambda/2-4\lambda+\lambda^2] + 1/2 - \lambda$

$= (2-\lambda)[\lambda^2 - 9/2\lambda + 2] - \lambda + 1/2$

$= 2\lambda^2 - 9\lambda + 4 - \lambda^3 + 9/2\lambda^2 - 2\lambda - \lambda + 1/2$

$= -\lambda^3 + \frac{13}{2}\lambda^2 - 12\lambda + 9/2$

$|A-\lambda I| = 0 \Rightarrow \lambda^3 - 13/2\lambda^2 + 12\lambda - 9/2 = 0$

$\lambda = 1/2, 3 \Rightarrow$ eigen values eigon values, $\lambda = 1, 4, 3$

? $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1/2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

A) $A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & -1 \\ 0 & 1/2-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{bmatrix}$

$|A - \lambda I| = (2-\lambda)[(1/2-\lambda)(4-\lambda)] - 1[-1/2+\lambda]$

$= (2-\lambda)[(2 - \lambda/2 - 4\lambda + \lambda^2] + 1/2 - \lambda$

$= (2-\lambda)[\lambda^2 - 9/2\lambda + 2] - \lambda + 1/2$

$= 2\lambda^2 - 9\lambda + 4 - \lambda^3 + 9/2\lambda^2 - 2\lambda - \lambda + 1/2$

$= -\lambda^3 + \frac{13}{2}\lambda^2 - 12\lambda + 9/2$

$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 13/2\lambda^2 + 12\lambda - 9/2 = 0$

rank $A - \lambda I < n$. $|A - \lambda I| = 0 \Rightarrow$ nonzero solution.

$\lambda = 1/2, 3 \Rightarrow$ eigon values

? $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & -\lambda \end{bmatrix}$

$|A - \lambda I| = 2-\lambda[(2-\lambda)-\lambda - 1] + 1[\lambda - 1] - 1[1 + 2 - \lambda]$

$= 2-\lambda[-2\lambda + \lambda^2 - 1] + \lambda - 1 - 1 - 2 + \lambda]$

$= 2-\lambda[\lambda^2 - 2\lambda - 1] + 2\lambda - 4]$

$= 2\lambda^2 - 4\lambda - 2 - \lambda^3 + 2\lambda^2 + \lambda + 2\lambda - 4$

$= \cancel{-\lambda^3 + 4\lambda^2 - 2\lambda - 6}$ $\quad -\lambda^3 + 4\lambda^2 - \lambda - 6$

$|A - \lambda I| = 0 \Rightarrow \cancel{\lambda^3 - 2\lambda^2 - 2\lambda + 6 = 0}$

$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$

$\lambda = -1, 3, 2$ // — eigon values

we've to find the eigen vectors corresponding to $\lambda = -1, 2, 3$.

case 1, $\lambda = -1$

$(A - \lambda I)x = 0$

$(A + I)x = 0$

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$3x_1 + -x_2 - x_3 = 0.$

$-x_1 + 3x_2 - x_3 = 0.$

$-x_1 - x_2 + x_3 = 0.$

solve using gausian elimination.

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$R_2 \to 3R_2 + R_1$

$R_3 \to 3R_3 + R_1.$

$$\begin{bmatrix} 3 & -1 & -1 \\ 0 & 8 & -4 \\ 0 & -4 & 2 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{2}R_2.$

$$\begin{bmatrix} 3 & -1 & -1 \\ 0 & 8 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$8x_2 - 4x_3 = 0$

$3x_1 - x_2 - x_3 = 0.$

$8x_2 = 4x_3$

$x_2 = x_3/2$

$3x_1 - 2x_2 - x_3 = 0$

$3x_1 - \frac{2x^3}{2} - x_3 = 0$

$3x_1 - \frac{x^3}{2} - x_3 = 0$

$3x_1 - \frac{3x_3}{2} = 0$

$3x_1 = \frac{2x_3}{2}$

$3x_1 = \frac{3x_3}{2}$

$x_1 = \frac{1}{2}x_3.$

$$X = \begin{bmatrix} \frac{1}{2}x_3 \\ x_3/2. \\ x_3 \end{bmatrix}$$

$x_2 = (1/2, 1/2, 1)$

case 2, $\lambda = 2.$

$(A - \lambda I)x = 0$

$(A - 2I)x = 0$

$$\begin{bmatrix} 0 & -1 & -1 \\ -3 & 0 & -1 \\ -1 & -1 & -2 \end{bmatrix} x = 0$$

solve using gausian elimenation.

$$\begin{bmatrix} 0 & -1 & -1 \\ -3 & 0 & -1 \\ -1 & -1 & -2 \end{bmatrix} \quad R_1 \leftrightarrow R_2.$$

$$\begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & -2 \end{bmatrix} \quad R_1 \to -1 \times R_1$$

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & -2 \end{bmatrix} \quad R_3 \to 3R_3 + R_1$$

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & -3 & -5 \end{bmatrix}$$ $R_2 \to -R_2$

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -3 & -5 \end{bmatrix}$$ $R_3 \to R_3 + 3R_2$.

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$3x_1 + x_3 = 0$ —(1)

$x_2 + x_3 = 0$ —(2).

$x_2 = -x_3$ ~~eqn (2)~~

$3x_1 = -x_3$

$x_1 = -\frac{x_3}{3}$

$$x = \begin{bmatrix} -x_3/3 \\ -x_3 \\ x_3 \end{bmatrix}$$

eigen vector $(-1/3, -1, 1)$

case 3, $\lambda = 3$.

$(A - 3I)\,x = 0$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -3 \end{bmatrix} x = 0$$ $R_2 \to R_2 - R_1$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -3 \end{bmatrix}$$ $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_2 \rightarrow R_2 - R_1$$

$\cancel{-x_1 - x_2 - x_3 = 0}$

$\cancel{-x_1 - x_2 - 3x_3 = 0}$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = 0$

$x_2 = k$

$x_1 = -x_2 - x_3$

$x_1 = -k$ //

$-x_1 - x_2 - x_3 = 0$

$-2x_3 = 0$

$x_3 = 0.$

$\cancel{x_2 = k}$, $X = \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix}$

$\underline{\underline{X = (-1, 1, 0)}}$

2 $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (2-\lambda)\left((2-\lambda)(1-\lambda)\right) - 1[1-\lambda] + 1[0]$$

$$= (2-\lambda)(2 - 2\lambda - \lambda + \lambda^2) - 1 + \lambda$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2) + \lambda - 1$$

$$= 2\lambda^2 - 6\lambda + 4 - \lambda^3 + 3\lambda^2 - 2\lambda + \lambda - 1$$

$$= -\lambda^3 + 5\lambda^2 - 7\lambda + 3.$$

$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$

$\lambda = 1, 1, 3$

Algebraic multiplicity of $1 \rightarrow 2$

" " of $3 \rightarrow 1$

$\lambda = 1$

$(A - \lambda I)X = 0$

$(A - I)X = 0.$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$x_3 = k_1, \; x_2 = k_2.$

$x_1 = -(k_1 + k_2).$

$$X = \begin{bmatrix} -(k_1+k_2) \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

no. of free variables $\to$ geometric multiplicity

If we have n geometric multiplicity then we have n eigen vectors. for eigen bases exist.

$$\left.\begin{array}{l} X_1 = (-1, 0, 1), \\ X_2 = (-1, 1, 0) \end{array}\right\} \text{eigen vectors.}$$

$\lambda = 3$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \quad R_3 \to -^{1} R_3$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

3×3 matrix
unknowns 3 generates

$-x_1 + x_2 + x_3 = 0.$

$2x_3 = 0$

$x_3 = 0$

$x_2 = K$

$x_1 = x_2 + x_3.$

$x_1 = K$

$$X = \begin{bmatrix} K \\ K \\ 0 \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

eigen vector $= (1,1,0)$

2. Find the algebraic and geometric multiplicity of the eigen values of the following matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 1 & -3 & 3-\lambda \end{bmatrix}$$

$$|A-\lambda I| = -\lambda[-\lambda(3-\lambda)+3] - 1[-1]$$

$$= -\lambda[-3\lambda+\lambda^2+3] + 1$$

$$= +3\lambda^2 - \lambda^3 - 3\lambda + 1$$

$$= -\lambda^3 + 3\lambda^2 - 3\lambda + 1$$

$$|A-\lambda I| = 0 \Rightarrow \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$\lambda = 1, 1, 1$

Algebraic multiplicity of r = 3.

$\lambda = 1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix} X = 0.$$

$R_3 \to R_3 + R_1$.

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$.

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$-x + y = 0$

$-y + z = 0.$

$x = y$

$y = z \Rightarrow x = y = z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \text{eigen vector}$$

It has no eigen basis

Eigen values of the transpose

The transpose A^T of a square matrix A has the same eigenvalues as A.

* The product of the characteristic roots (eigen values) of a square matrix of order n is equal to the determinant of the matrix.

* For a square matrix A, λ is a eigen value iff there exists a non zero vector x such that $AX=\lambda X$

Eigen Basis

* An eigenbasis is a basis consisting of eigen vectors of A

* Eigenvectors with different eigenvalues are automatically linearly independent.

* If an $n \times n$ matrix A has n distinct eigenvalues, then it has an eigen basis.

* A matrix possesses inverse iff all of its eigen values are nonzero.

* Let us consider a $(n \times n)$ matrix A, whose eigen values

are $\lambda_1, \lambda_2, \dots \lambda_n$

then,

a) Trace of matrix A is equal to sum of its eigen values. ie,

$$tr(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

b) $\det(A) = \lambda_1 \cdot \lambda_2 \dots \lambda_n$

c) Eigen values of k^{th} power of matrix A ie, A^k will be $\lambda_1^k, \lambda_2^k, \dots \lambda_n^k$ (eg. eigen value of $A^2 = \lambda_1^2, \lambda_2^2 \dots \lambda_n^2$)

d) If the matrix A is invertible, then its inverse A^{-1} does have eigen values $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots \frac{1}{\lambda_n}$.

Q. S.T the eigen value for A^2 is λ^2.

A). suppose λ is the eigen value of A then,

$$AX = \lambda X \quad \text{---} (1)$$

multiply with A on (1)

$$A^2X = \lambda(AX)$$

$$A^2X = \lambda(\lambda X) \quad (\text{by } (1))$$

$$A^2X = \lambda^2 X$$

$\Rightarrow$ eigen value of A^2 is λ^2

Q. The product of 2 eigen values of the matrix is 16, Find the third eigen value.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

A) By the theorem, $\det(A) = \lambda_1 \lambda_2 \lambda_3$

$$\det(A) = 16 \cdot \lambda_3$$

$|A| = 6(9-1) + 2(-6+2) + 2(2-6)$

$= 6\times 8 + 2\times -4 + 2\times -4$

$= 48 - 8 - 8$

$= 32.$

$32 = 16\lambda_3$

$\lambda_3 = 2$ //

$\lambda = 2$ is the third eigen value.

2 Find the constants a, b such that the matrix $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ has 3, −2 as its eigen value.

A). By the theorem, $tr(A) = \lambda_1 + \lambda_2$.

$a + b = 1$ —— (1).

By the theorem, $\det(A) = \lambda_1 \lambda_2$.

$ab - 4 = -6$

$ab = -2$ —— (2).

$b = -2/a$. —— (3)

sub (3) in (1)

$a - 2/a = 1$

$a^2 - 2 = a$

$a^2 - a - 2 = 0$

$(a+1)(a-2)$

$a = 2$ or $a = -1$.

when $a = 2$, $b = -1$ when $a = -1$, $b = 2$.

Diagonalisation

find A^{10}

$A^2 = XDX^{-1}XDX^{-1}$

$= XD^2X^{-1}$

$A^{10} = XD^{10}X^{-1}$

diagonalise ചെയ്യാൻ പറ്റണം എങ്കിൽ 3x3 മാട്രിക്സ്-ന് 3 eigen vector വേണം.

$A = XDX^{-1}$ $X^{-1}AX = D$

2 diagonalize the matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

A) Find eigen values,

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix}$$

$= (1-\lambda)((3-\lambda)^2 - 1)$

$= (1-\lambda)(9 - 6\lambda + \lambda^2 - 1)$

$= (1-\lambda)(\lambda^2 - 6\lambda + 8)$

$= \cancel{\lambda^2 - 6\lambda + 8 - \lambda^3 + 6\lambda^2 - 8\lambda}$

$= \cancel{-\lambda^3 + 7\lambda^2}$

$P = 1$

$S = -6$

$\lambda = 1$ or $\lambda^2 - 6\lambda + 8 = 0$

$(\lambda-2)(\lambda-4)$

$\lambda = 1, 2, 4$

Eigen vector for 1

$(A - I)x = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} x = 0.$$

$\cancel{2x_2}$ $R_3 \to 2R_3 + R_2.$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$x_3 = 0$

$2x_2 = 0$

$x_2 = 0.$, here we don't know the value of x_1 so

x_1 is arbitrary

$$x = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix}$$

in particular $x(1, 0, 0)$

$\lambda = 2.$

$(A - 2I)x = 0$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} x = 0 \qquad R_3 \to R_3 + R_2.$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = K$

$x_2 + -x_3 = 0 \Rightarrow x_2 - K = 0$

$x_2 = K.$

$x_1 = 0.$

$$x = \begin{bmatrix} 0 \\ K \\ K \end{bmatrix} = (0, 1, 1)$$

$\underline{\lambda = 4}$

$(A - 4I)x = 0$

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} x = 0. \qquad R_3 \to R_3 - R_2.$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$-3x_1 = 0$

$x_1 = 0.$

$-x_2 - x_3 = 0.$

$x_3 = K$

$-x_2 = K$

$x_2 = -K.$

$$x = \begin{bmatrix} 0 \\ -K \\ K \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} //$$

$\therefore$

$X^{-1}AX = D$

where $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$X^{-1} =$

~~hence $X^{-1}AX = D$~~

* Note:

If an $n \times n$ matrix A has n distinct eigen values, then A has a basis of eigenvectors $x_1, x_2, \ldots, x_n$ for R^n.

* Even if not all n eigen values are different, a matrix A may still provide an eigen basis for R^n.

* A may not have enough linearly independent eigenvectors to make up a basis.

* A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

Similarity of matrices, Diagonalization.

* Eigenbases also play a role in reducing a matrix A to a diagonal matrix whose entries are the eigen values of A. This is done by a "similarity transformation".

* An $n \times n$ matrix $\hat{A}$ is called similar to an $n \times n$ matrix A if $\hat{A} = P^{-1}AP$ for some (nonsingular) $n \times n$ matrix P. This transformation, which gives $\hat{A}$ from A, is called a similarity transformation.

Eigenvalues and Eigenvectors of similar matrices.

* If $\hat{A}$ is similar to A, then $\hat{A}$ has the same eigen values as A.

* Furthermore, if x is an eigen vector of A, then $y = P^{-1}x$ is an eigen vector of $\hat{A}$ corresponding to the same eigenvalue.

Diagonalization of a matrix.

If an nxn matrix A has a basis of eigenvectors, then $D = X^{-1}AX$ is diagonal, with the eigen values of A as the entries on the main diagonal. Here X is the matrix with these eigen vectors as column vectors. Also

$$D^m = X^{-1}A^m X$$

eg:
$$D^2 = DD = (X^{-1}AX)(X^{-1}AX)$$
$$= X^{-1}A(XX^{-1})AX$$
$$= X^{-1}AAX$$
$$= X^{-1}A^2X$$

?. Diagonalize

$$A = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$$

A) Find eigen values of A

$$(A-\lambda I) = \begin{bmatrix} 7.3-\lambda & 0.2 & -3.7 \\ -11.5 & 1.0-\lambda & 5.5 \\ 17.7 & 1.8 & -9.3-\lambda \end{bmatrix}$$

$$|A-\lambda I| = (7.3-\lambda)\left[(1-\lambda)(-9.3-\lambda) - (5.5\times 1.8)\right] - 0.2\left[11.5\times(9.3+\lambda) - (17.7\times 5.5)\right.$$

2. check whether the following matrix can be diagonalize or not.

1. $A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$A-\lambda I = \begin{bmatrix} 2-\lambda & -1 & 3 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$

$|A-\lambda I| = (2-\lambda)\left[(2-\lambda)(3-\lambda)-1\right]+1\left[-3+\lambda+1\right]+3\left[1-2+\lambda\right]$

$= (2-\lambda)(6-2\lambda-3\lambda+\lambda^2-1] + \lambda-2 + 3[\lambda-1]$

$= (2-\lambda)(\lambda^2-5\lambda+5)+\lambda-2+3\lambda-3$

$= 2\lambda^2-10\lambda+10-\lambda^3+5\lambda^2-5\lambda+\lambda-2+3\lambda-3$

$= -\lambda^3+7\lambda^2-11\lambda+5$

$=$

$= \lambda^3-7\lambda^2+11\lambda-5$

$\lambda=1 \Rightarrow$

$1-7+14-8$

$-6+14-7$

$=0$

$\lambda=1$ is solution

$(\lambda-1)(\lambda^2-6\lambda+5)$

$(\lambda-1)(\lambda-1)(\lambda-5)$

$P=-8$

$S=-6$

$$\begin{array}{r|l} & \lambda^2-6\lambda+5 \\ \hline \lambda-1 & \lambda^3-7\lambda^2+11\lambda-5 \\ & \lambda^3-\lambda^2 \\ \hline & -6\lambda^2+11\lambda \\ & -6\lambda^2+6\lambda \\ \hline & 5\lambda-5 \\ & 5\lambda-5 \\ \hline & 0 \end{array}$$

$\lambda = 1, 5, 1$

$\lambda = 1$

$$\begin{bmatrix} 1 & -1 & 3 \\ -1 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = 0$

$x_1 - x_2 = 0$

$x_1 = x_2.$

$(x_2 = k)$

$x_1 = k.$

$$x = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = (1, 1, 0)$$

since there is only one linearly independent eigen vector corresponding to $\lambda = 1$ so it is not diagonalizable.

$\lambda = 5$

$$\begin{bmatrix} -3 & -1 & 3 \\ -1 & -3 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

2 Find diagonalize the matrix and hence find A^6.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

$$=(1-\lambda)((5-\lambda)(1-\lambda)-1)-1[(1-\lambda)-3]+3(1-15+3\lambda)$$

$$=(1-\lambda)[5-5\lambda-\lambda+\lambda^2-1]-[-2-\lambda]+3(-14+3\lambda)$$

$$=(1-\lambda)(\lambda^2-6\lambda+4)+\lambda+2+9\lambda-42$$

$$=\lambda^2-6\lambda+4-\lambda^3+6\lambda^2-4\lambda+\lambda+2+9\lambda-42$$

$$=-\lambda^3+7\lambda^2-36$$

$$\lambda^3-7\lambda^2+36=0$$

$\lambda=2$,

$8-7\times4+36$

$=8-28+36\neq0$

$\lambda=-2$,

$-8-28+36=0$

$\lambda=-2$ is a solution,

$$\begin{array}{r} \lambda^2-9\lambda+18 \\ \lambda+2 \enclose{longdiv}{\lambda^3-7\lambda^2+36} \\ \underline{\lambda^3+2\lambda^2} \\ -9\lambda^2+0\lambda \\ \underline{-9\lambda^2-18\lambda} \\ 18\lambda+36 \\ \underline{18\lambda+36} \\ 00 \end{array}$$

$(\lambda+2)(\lambda^2-9\lambda+18)=0$

$(\lambda+2)(\lambda-6)(\lambda-3)=0$

$\lambda=-2, \lambda=6, \lambda=3$

$\underline{\lambda=-2}$

$$A+2I=\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

$R_2 \to 3R_2-R_1$

$R_3 \to R_3-R_1$

$$\sim \begin{bmatrix} 3 & 1 & 3 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$3x_1 + x_2 + 3x_3 = 0 \rightarrow 3x_1 = -3K$

$x_1 = -K$.

$2x_2 = 0$

$x_3 = K$.

$$X = \begin{bmatrix} -K \\ 0 \\ K \end{bmatrix} \quad (-1, 0, 1)$$

$\lambda = 6$

$$A - 6I = \begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}$$

$R_2 \rightarrow 5R_2 + R_1$

$R_3 \rightarrow 5R_3 + 3R_1$

$$\sim \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 8 & -16 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 2R_2$

$$\sim \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$-4x_2 = -8K$

$\lambda = -2$

$(-1, 0, 1)$

$\lambda = 6.$

$-5x_1 + x_2 + 3x_3 = 0$

$-4x_2 + 8x_3 = 0.$

$x_3 = K$

$-4x_2 + 8K = 0.$

$-4x_2 = -8K$

$x_2 = 2K.$

$-5x_1 + 2K + 3K = 0.$

$-5x_1 + 5K = 0$

$-5x_1 = -5K$

$x_1 = K$

$(1, 2, 1)$

$\lambda = 8:$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$

$R_3 \to 2R_3 + 3R_1$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = K$

$5x_2 + 5x_3 = 0$

$-2x_1 + x_2 + 3x_3 = 0$

$5x_2 + 5K = 0$

$5x_2 = -5K$

$x_2 = -K$

$-2x_1 - K + 3K = 0$

$-2x_1 + 2K = 0$

$-2x_1 = -2K$

$x_1 = K$

$(1, -1, 1)$

$$X = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$D = X^{-1} A X$

where $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$A = X D X^{-1}$

$A^6 = X D^6 X^{-1}$

$$= X \begin{bmatrix} -2^6 & 0 & 0 \\ 0 & 6^3 & 0 \\ 0 & 0 & 3^6 \end{bmatrix} X^{-1}$$

Inverse

1. Gauss-Jordan elimination

$(AI =)$

2. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $A^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

3.

Quadratic forms, Transformation to principal axes

By definition, a quadratic form Q in the components $x_1, x_2, \ldots, x_n$ of a vector x is a sum of n^2 terms, namely,

$$Q = x^T A x = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n$$
$$+ a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n$$
$$+ \cdots \cdots$$
$$+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2$$

$A = [a_{jk}]$ is called the coefficient matrix of the form. we may assume that A is symmetric.

* A quadratic form is a polynomial with terms all of degree two.

Transformation to principal Axes, conic sections

Q. $17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$, transform the form to canonical form and find the conic section.

A Form matrix

$$A = \begin{array}{c|cc} & x_1 & x_2 \\ \hline x_1 & 17 & -15 \\ x_2 & -15 & 17 \end{array}$$

$|A - \lambda I| = (17-\lambda)^2 - 15^2 =$

$(17-\lambda)^2 = 225$

$17-\lambda = \pm 15$

$\lambda = \pm 15 + 17$

$\lambda = 2, 32$

$Q = 2y_1^2 + 32y_2^2 \longrightarrow$ canonical form.

~~eigen vector~~

we see that $Q = 128$ represents the ellipse

$2y_1^2 + 32y_2^2 = 128$ ie,

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1.$$

If we want to know the direction of the principal axes in the x_1x_2 - coordinates, we have to determine normalized eigenvectors from $(A-\lambda I)x=0$ with $\lambda = 2$, and $\lambda = 32$.

when $\lambda = 2$,

$$A - \lambda I = \begin{bmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{bmatrix} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$(A-\lambda I)x = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} x$$

$$= \begin{bmatrix} 15 & -15 \\ 0 & 0 \end{bmatrix} x \qquad R_2 \to R_2 + R_1$$

$x_2 = K$

$15x_1 - 15K = 0$

$15x_1 = 15K$

$x_1 = K.$

$(1,1) \to \qquad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

To normalize the eigen vector divide it with magnitude

|e|

$|x| = \sqrt{a^2+b^2}$

$= \sqrt{1+1} = \sqrt{2}.$

normalized eigen vector = $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ ——— (1)

when $\lambda = 32.$

$(A - 32I)x = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$ $\quad R_2 \to R_2 - R_1$

$= \begin{bmatrix} -15 & -15 \\ 0 & 0 \end{bmatrix}$

$x_2 = K.$

$-15x_1 - 15K = 0.$

$-15x_1 = 15K$

$x_1 = -K.$

$X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

normalized eigen vector = $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ ——— (2).

from (1) and (2)

$x = Xy = \begin{bmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$x_1 = 1/\sqrt{2}\, y_1 - 1/\sqrt{2}\, y_2$

$x_2 = 1/\sqrt{2}\, y_1 + 1/\sqrt{2}\, y_2.$

2) $x_1^2 - x_1x_2 + x_2^2 = 8$

A matrix form.

$A = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$

$|A-\lambda I| = 0 \Rightarrow$

$$(1-\lambda)^2 - 1/4 = 0.$$

$$(1-\lambda)^2 = 1/4$$

$$1-\lambda = \pm 1/2$$

$$\lambda = 1 \pm 1/2$$

$$\lambda = 3/2 , 1/2.$$

canonical form.

$$1/2\, y_1^2 + 3/2\, y_2^2 = 8$$

The conic section is ellipse.

$\lambda = 1/2$

$$(A-\lambda I)x = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1/2 & -1/2 \\ 0 & 0 \end{bmatrix} \qquad R_2 \to R_2 + R_1$$

$$x_2 = k,$$

$$1/2\, x_1 = 1/2\, x_2$$

$$x_1 = k$$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{---} (1)$$

$\lambda = 3/2$

$$(A - 3/2\lambda)x = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix} \qquad R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} -1/2 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$$x_2 = k, \quad x_1 = -k$$

$$X = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{——— (2)}$$

From (1) & (2).

$$x = Xy = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}.$$

principal Axes theorem

The substitution $x = Xy$ transforms a quadratic form $Q = x^T A x = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} x_j x_k \quad (a_{kj} = a_{jk})$ to the principal axes form or canonical form

$$Q = y^T D y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

where $\lambda_1, \dots, \lambda_n$ are the (not necessarily distinct) eigenvalues of the (symmetric) matrix A, and X is an orthogonal matrix with corresponding eigenvectors $x_1, x_2, \dots, x_n$ respectively, as column vectors.

Note:

circle: when x and y are both squared and the coefficients on them are the same including the sign. eg: $3x^2 - 12x + 3y^2 = 2$.

parabola: when either x or y is squared - not both. eg: $y = x^2 - 4$ and $x = 2y^2 - 3y + 10$

Ellipse: when x and y are both squared and the coefficients are +ve but different

eg: $3x^2 - 9x + 8y^2 + 10y - 6 = 0$

Hyperbola: when x and y are both squared, and exactly one of the coefficients is negative and exactly one of the coefficients is positive

eg: $4y^2 - 10y - 3x^2 = 12$.

Fundamental Theorem For linear systems

(a) Existence: A linear system of m equations in n unknowns $x_1, \dots, x_n$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \quad \text{—(1)}$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

is consistent, ie, has solutions, iff the coefficient matrix A and the augmented matrix $\tilde{A}$ have the same rank. Here.

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ \vdots & \dots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \text{ and } \tilde{A} \begin{bmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \dots & \vdots & \vdots \\ \vdots & \dots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{bmatrix}$$

(b) Uniqueness: The system (1) has precisely one solution iff this common rank r of A and $\tilde{A}$ equals n.

c). Infinitely many solutions: If this common rank r

is less than n, the system (1) has infinitely many solutions. All of these solutions are obtained by determining r suitable unknowns (whose submatrix of coefficients must have rank r) in terms of the remaining n-r unknowns, to which arbitrary values can be assigned.

d) Gauss elimination: If solution exist, they can all be obtained by the Gauss elimination.